

Quantum distributions prescribed by factorization hypothesis of probability

Qiuping A. Wang

*Institut Supérieur des Matériaux du Mans,
44, Avenue F.A. Bartholdi, 72000 Le Mans, France
(February 1, 2008)*

Nonextensive quantum gas distributions are investigated on the basis of the factorization hypothesis of compound probability required by thermodynamic equilibrium. It is shown that the formalisms of Tsallis nonextensive statistical mechanics with normalized average give distribution functions for standard bosons and fermions obeying Pauli principle. The formalism with unnormalized average leads to a intermediate quantum distribution comparable to that of fractional exclusion statistics, with Fermi surface at $T = 0$ depending on the parameter q .

05.20.-y;05.30.-d;05.30.Pr

I. INTRODUCTION

It is believed that long-range correlations can often (if not always) be related to fractal or chaotic space-time structures and also to power laws and nonextensive phenomena [1–6]. That is why the fractal inspired nonextensive statistical mechanics (*NSM*) [7] is receiving much attention and widely considered as a possible valid theory to describe systems with complex correlations. Considering the fractal or chaotic behaviors and nonextensive or fractional effects in quantum systems like, for example, correlated electrons in superconductor, solitons and quasi-particles in condensed matters [1,8–11], we naturally expect a quantum statistical mechanics within *NSM*. One of the first steps in this direction is taken by Büyükkiliç et al [12] in giving the generalized quantum distributions with the so called factorization approximation.

The derivation of the conventional quantum one-body distribution from the compound many-body distribution is straightforward within Boltzmann-Gibbs statistics because the latter considers only short range interactions with additive entropy and energy. This allows the many-body density operator ρ to be factorized into the product of one-body density operators ρ_i of N particles of the system :

$$\rho = \prod_{i=1}^N \rho_i. \quad (1)$$

But in nonextensive statistical mechanics, this kind of factorization of compound probability is impossible if we still consider, for independent particles or ideal gas, extensive energy with total hamiltonian H given by

$$H = \sum_{i=1}^N H_i = \sum_{i=1}^N \left(\frac{p_i^2}{2m} + V_i \right), \quad (2)$$

where $H_i = \frac{p_i^2}{2m} + V_i$ is the one-body hamiltonian of i^{th} particle, p_i its momentum and V_i its potential energy. To overcome this difficulty, a so called factorization approximation is proposed [12] which imposes Eq.(1) in keeping Eq.(2). This approximation allows to obtain a nonextensive quantum distribution of *NSM* with complete distribution (i.e. $\text{Tr} \rho = 1$) and unnormalized average (i.e. $\bar{x} = \text{Tr} \rho^q x$ where x is certain operator [7]). It is, explicitly or not, employed in most of the applications of *NSM* to the cases like, among others, boson and fermion systems [12–14], the polytropic model of galaxies, solar neutrinos, peculiar velocity of galaxy clusters, electron plasma turbulence (for updated comments on these works, we refer to reference [15]), and the application of nonextensive blackbody distribution to laser physics [16]. Although these applications clearly evidence the existence and the important role of the q -exponential distribution in the nature, the justification of the passage from many body q -distribution to one-body one is missing. Recently, a direct analysis [17] shows that, assuming Eq.(1) and Eq.(2) at the same time means neglecting a correlation energy which is in general not negligible. A detailed computation for N -oscillator system [18] also shows that the partition function given on the basis of Eq.(1) is completely different from that given by using Eq.(2).

In this paper, the factorization hypothesis is discussed from a new point of view relative to Abe's general pseudoadditivity [21] for nonextensive systems in thermal equilibrium. This discussion leads to the conclusion that Eq.(1) must be respected in all coherent and exact treatments of equilibrium systems. On this basis, quantum distributions

are derived within different formalisms of NSM : 1) complete distribution ($\text{Tr}\rho = 1$) with the normalized average $\bar{x} = \text{Tr}\rho x$ [7]; 2) incomplete distribution with q -normalization $\text{Tr}\rho^q = 1$ and the normalized average $\bar{x} = \text{Tr}\rho^q x$ [17]; 3) complete distribution with unnormalized average $\bar{x} = \text{Tr}\rho^q x$ [7].

II. THERMAL EQUILIBRIUM AND FACTORIZATION OF COMPOUND PROBABILITY

In Boltzmann-Gibbs statistics, Eq.(1) is a natural result of the short-range interactions or of the independence of the subsystems which necessarily lead to extensive entropy S and internal energy E . With the temperature definition $1/T = \beta = \frac{\partial S}{\partial E}$ (Boltzmann constant $k=1$), the extensivity of S and E allows to verify the zeroth law or the existence of thermal equilibrium. So, the uniqueness of Gibbs-Shannon entropy $S = -\sum_i p_i \ln p_i$ subject to Shannon-Khinchin axioms [19] can be regarded from a different viewpoint. We can also state that, given Gibbs-Shannon entropy, the existence of thermal equilibrium necessarily leads to the additivity of entropy or to Eq.(1). This statement seems self-evident and useless for Boltzmann-Gibbs statistical mechanics, because Eq.(1) and the axiom about the additivity of entropy are obvious in view of the independence of subsystems. But it is not the case in NSM . The pseudoadditivity axiom $S = S_1 + S_2 + (1-q)S_1S_2$ [7] for a composite system containing two subsystems 1 and 2 with respectively entropies S_1 and S_2 , is in fact a consequence of Eq.(1) supposed Tsallis entropy $S = -\frac{1-\text{Tr}\rho^q}{1-q}$. As a consequence, the nonextensive theory is entirely subject to the “independence” of subsystems. The problem is that, in nonextensive physics, the subsystems are in general dependent on each other. So Eq.(1) or the pseudoadditivity is in this case somewhat artificial and seems to hide something which may restrict the validity of the theory. In many applications of NSM , Eq.(1) is sometimes respected, sometimes not, sometimes combined with the incompatible extensive hamiltonian Eq.(2) [12,18,20]. This confused situation is in our opinion due to the fact that Eq.(1) is not clearly founded for nonextensive statistics subject to Tsallis entropy.

Very recently, Abe [21] proposed a general pseudoadditivity for entropy required by the existence of thermal equilibrium in composite nonextensive systems : $f(S) = f(S_1) + f(S_2) + \lambda f(S_1)f(S_2)$ where f is certain differentiable function satisfying $f(0) = 0$ and λ a constant depending on the nature of the system of interest. So for a system containing N subsystems, the thermal equilibrium requires following additivity :

$$\ln[1 + \lambda f(S)] = \sum_{i=1}^N \ln[1 + \lambda f(S_i)]. \quad (3)$$

On the other hand, Eq.(3) applied to Tsallis entropy means $f(S) = S$ and $\lambda = 1 - q$ [21], which directly leads to $\ln \text{Tr}\rho^q = \sum_{i=1}^N \ln \text{Tr}\rho_i^q$ or Eq.(1). This interesting result raises Eq.(1) from the level of a special assumption for “statistically independent” subsystems to the level a general theorem of equilibrium thermodynamics with either Boltzmann-Gibbs or Tsallis entropy. We do not need any more the independence of subsystems to write Eq.(1) which must be respected by all probabilities of the systems (extensive or nonextensive) that can have thermal equilibrium. Consequently, all calculations based on the factorization theorem or using one-body q -exponential distribution are, as a matter of fact, correct and exact applications of NSM . And all calculations of NSM based on Eq.(2) [not compatible with Eq.(1)] should now be regarded as “extensive energy approximation” which may be very different from the exact treatment and should be employed with great care.

In what follows, we discuss the quantum distributions proposed by Büyükkiliç *et al* [12] without any approximation. On the basis of the factorization theorem discussed above, the derivation of this kind of distribution is straightforward.

III. COMPLETE DISTRIBUTION FORMALISM

Let us begin with the complete distribution ($\text{Tr}\rho = 1$) and the normalized average $\bar{x} = \text{Tr}\rho x$. This formalism of NSM is first proposed by Tsallis [7] and has received little attention due to some problems [7] (e.g. missing Legendre transformation and zeroth law, thermodynamic stability problem discussed by assuming extensive energy [23]). Since several years, NSM has much evolved. Legendre transformation and the zeroth law in this formalism can be established by using the method of the references [17]. The stability problem must be revisited with nonextensive energy satisfying Eq.(1). So this standard formalism of probability theory remains a possible choice of NSM and still deserves to be studied. The reader will see below that the formalisms with normalized average can give the very distribution functions of Büyükkiliç *et al*.

For nonextensive quantum gas, we have [7,12,22] :

$$\rho = \frac{1}{Z} [1 - (q-1)\beta(H - \mu N)]^{\frac{1}{q-1}} = \frac{1}{Z} e^{-\beta h'} \quad (4)$$

where $h' = \frac{\ln[1+(1-q)\beta(H-\mu N)]}{\beta(1-q)}$ can be called “deformed hamiltonian”. The grand partition function Z is then given by

$$\begin{aligned} Z &= \text{Tr}[1 - (q-1)\beta(H - \mu N)]^{\frac{1}{q-1}} \\ &= \text{Tr}e^{-\beta h'}. \end{aligned} \quad (5)$$

From the factorization theorem, we can write

$$\begin{aligned} Z &= \text{Tr}e^{-\beta \sum_n h'_i} \\ &= \prod_k \sum_{n_k} e^{-n_k \beta \epsilon_k} \\ &= \prod_k \sum_{n_k} e^{-\beta n_k \epsilon_k} \end{aligned} \quad (6)$$

where ϵ_k is the eigenvalue of $h'_i = \frac{\ln[1+(1-q)\beta(H_i-\mu)]}{\beta(1-q)}$, the deformed one-particle hamiltonian satisfying $h' = \sum_{i=1}^N h'_i$, and n_k the occupation number of the one-particle state k . For boson and fermion, we obtain, respectively,

$$Z = \prod_k \sum_{n_k=0}^{\infty} e^{-n_k \beta \epsilon_k} = \prod_k \frac{1}{1 - e^{-\beta \epsilon_k}} \quad \text{and} \quad Z = \prod_k \sum_{n_k=0}^1 e^{-n_k \beta \epsilon_k} = \prod_k (1 + e^{-\beta \epsilon_k}). \quad (7)$$

Then, it is straightforward to show that, just as in the conventional quantum statistics ,

$$\bar{n}_l = \text{Tr} \rho n_l = -\frac{1}{\beta} \frac{\partial(\ln Z)}{\partial \epsilon_l} = \frac{1}{e^{\beta \epsilon_l} \pm 1} = \frac{1}{[1 + (1-q)\beta(e_l - \mu)]^{\frac{1}{1-q}} \pm 1} \quad (8)$$

where e_l is the eigenvalue of the one-particle hamiltonian H_i . “+” and “-” correspond to fermion and boson, respectively. This result is just that given by Büyükkiliç *et al* [12] with only a change $(q-1) \rightarrow (1-q)$.

IV. INCOMPLETE DISTRIBUTION FORMALISM

For incomplete distribution with the q -normalization, the Grand partition function Z is given by

$$\begin{aligned} Z^q &= \text{Tr}[1 - (1-q)\beta(H - \mu N)]^{\frac{q}{1-q}} \\ &= \text{Tr}e^{-q\beta h'} \\ &= \text{Tr}e^{-q\beta \sum_n h'_i} \\ &= \prod_k \sum_{n_k} e^{-q n_k \beta \epsilon_k} \\ &= \prod_k \sum_{n_k} e^{-q \beta n_k \epsilon_k}. \end{aligned} \quad (9)$$

The deformed hamiltonians h' and h'_i are the same functions as given above with the transform $(q-1) \rightarrow (1-q)$ [17]. The same machinery leads to

$$\bar{n}_l = \text{Tr} \rho^q n_l = \frac{1}{e^{q\beta \epsilon_l} \pm 1} = \frac{1}{[1 + (q-1)\beta(e_l - \mu)]^{\frac{q}{q-1}} \pm 1}. \quad (10)$$

These distributions are equivalent to Eq.(8) with only a difference in the q -dependence. Eq.(8) and Eq.(10) represent standard bosons and fermions satisfying Pauli exclusion principle. For example, $n_l \leq 1$ for fermions. It should be noticed that, in this case, the Fermi surface at $T = 0$ or $\beta = \infty$ is the same as in the conventional case and independent of the parameter q .

V. COMPLETE DISTRIBUTION WITH UNNORMALIZED AVERAGE

On the other hand, in the formalism of complete distribution with unnormalized average, the things will be different because this average, unlike the normalized ones, can lead to nonadditive deformed internal energy even with additive deformed hamiltonian h' . This formalism received a lot of attention and finally was found to show some puzzling properties [7]. But up to now, there is no really solid reason for abandoning it definitely. We still see various kind of unnormalized expectations widely used in many fields, especially in financial problems [6,24]. It is also found that this formalism is the only one of *NSM* that can give standard statistical interpretation of heat and work [25]. So it is still of interest to study this formalism in the circumstance of quantum systems.

The partition function is still given by Eq.(5) with a transform $(q-1) \rightarrow (1-q)$ [7]. The occupation number is calculated as follows

$$\begin{aligned}
\bar{n}_l &= \text{Tr} \rho^q n_l \\
&= \text{Tr} n_l \frac{1}{Z^q} [1 + (q-1)\beta(H - \mu N)]^{\frac{q}{1-q}} \\
&= \frac{1}{Z^q} \prod_k \sum_{n_k} n_l e^{-qn_k \beta \epsilon_k} \\
&= \frac{-1}{q\beta Z^q} \frac{\partial}{\partial \epsilon_l} \prod_k \sum_{n_k} e^{-qn_k \beta \epsilon_k} \\
&= -\frac{1}{Z^q} \prod_k \sum_{n_k} e^{-qn_k \beta \epsilon_k} \frac{1}{e^{q\beta \epsilon_l} - 1} \\
&= \text{Tr} \rho^q \frac{1}{e^{q\beta \epsilon_l} - 1} \\
&= \frac{1}{e^{q\beta \epsilon_l} - 1} \prod_k \frac{(1 - e^{-\beta \epsilon_k})^q}{1 - e^{-q\beta \epsilon_k}}
\end{aligned} \tag{11}$$

for bosons and

$$\bar{n}_l = \frac{1}{e^{q\beta \epsilon_l} + 1} \prod_k \frac{1 + e^{-q\beta \epsilon_k}}{(1 + e^{-\beta \epsilon_k})^q} \tag{12}$$

for fermions. Eq.(11) and (12) can be written as

$$\bar{n}_l = \frac{Q_{\pm}}{e^{q\beta \epsilon_l} \pm 1} = \frac{Q_{\pm}}{[1 + (q-1)\beta(e_l - \mu)]^{\frac{q}{q-1} \pm 1}} \tag{13}$$

where $Q_+ = \text{Tr} \rho^q = \prod_k \frac{1+e^{-q\beta \epsilon_k}}{(1+e^{-\beta \epsilon_k})^q}$ (or $Q_- = \prod_k \frac{(1-e^{-\beta \epsilon_k})^q}{1-e^{-q\beta \epsilon_k}}$) can be regarded as a parameter depending on q . $Q_{\pm} > 1$, $Q_{\pm} = 1$ and $Q_{\pm} < 1$ for $q < 1$, $q = 1$ and $q > 1$, respectively. These distributions seems interesting because they allow intermediate occupation number between that of bosons and fermions. In particular, at absolute zero, for “fermionlike” particle with “+” in Eq.(13), $\bar{n}_l = Q$ when $\epsilon_l < \mu$ and $\bar{n}_l = 0$ when $\epsilon_l > \mu$. This means that it is possible for several “fermions” to occupy an one-particle quantum state if $Q > 1$ or $q < 1$. Consequently, the Fermi surface ϵ_F at $T = 0$ or $\beta = \infty$ changes as a function of Q or of the interaction between the particles : $\epsilon_F = \frac{\epsilon_F}{Q^{2/3}}$ where ϵ_F is the conventional Fermi energy at $q = 1$ or $Q = 1$. This result can be compared to that of the *fractional exclusion statistics* [9] for intermediate particles different from bosons and fermions. It is not surprising to see that nonextensive statistics has similar effect to that of fractional statistics describing interacting particle or elementary excitation [9,10], because the q -distribution is nothing but a result of long range interactions. However, the fact that only the *NSM* formalism with unnormalized average can give the intermediate quantum distributions seems to deserve further investigation. Quite interesting efforts have been made by some authors [14,26] to relate nonextensive statistics and the quantum distributions given by Eq.(8) or (10) to fractional exclusion statistics.

It is obvious that if we assume the q -normalization for incomplete distribution, $Q = 1$ and Eq.(13) will become Eq.(10).

VI. CONCLUSION

Summing up, on the basis of the factorization theorem of compound probability prescribed by thermodynamic equilibrium, the quantum gas distributions within different formalisms of *NSM* are investigated. It is shown that only the formalisms with normalized average can give distribution functions for standard bosons and fermions. The formalism with unnormalized average leads to a intermediate distribution similar to that of fractional exclusion statistics, with Fermi surface at $T = 0$ depending on the parameter q . This different quantum properties of *NSM* due to normalizations remains something to be understood. A detailed study of the relation between the unnormalized quantum distribution and the fractional exclusion statistics would be of interest.

-
- [1] Kröger H, Fractal geometry in quantum mechanics field theory and spin systems, *Physics Report*, 2000;323(2):82-181
 - [2] El Naschie MS, On the electroweak mixing angle in E^∞ *Chaos, Solitons & Fractals*, 2000;11(11):1803-1807
 - [3] Nottale L, Scale relativity and fractal space-time *Chaos, Solitons & Fractals*, 1966;7(6):877-938
 - [4] Ahmed E, Hegazi AS, On infinitesimally deformed algebra and fractal space-time theory *Chaos, Solitons & Fractals*, 2000;11(11):1759
 - [5] Mandelbrot B., *Fractals in Finance*, Springer, Berlin, 1997
 - [6] C. Tsallis, *Chaos, Solitons & Fractals*, 1995;6:539.
 - [7] Tsallis C., *J. Statis. Phys.*, 1988;52:479; Curado E.M.F., and Tsallis C., *J. Phys. A: Math. Gen.* 1991;24:L69; Tsallis C, Mendes R.S. and Plastino A.R., *Physica A*, 1998;261:534
 - [8] Badiali J.P., *Phys. Rev. E*, 1999;60:2533
 - [9] Wu Y.S., *Phys. Rev. Lett.*, 1994;73:922
 - [10] Haldane F.D.M., *Phys. Rev. Lett.*, 1991;67:937
 - [11] Monteiro MR, Roditi I., *Mod. Phys. Lett. B*, 1993;7:1843
 - [12] Büyükkiliç F. and Demirhan D., *Phys. Lett. A*, 1993;181:24; Büyükkiliç F., Demirhan D. and Güleç A., *Phys. Lett. A*, 1995;197:209; Tirnakli U., Büyükkiliç F. and Demirhan D., *Physica A*, 1997;240:657; Tirnakli U., Büyükkiliç F. and Demirhan D., *Phys. Lett. A*, 1998;245:62; Torres D.F. and Tirnakli U., *Physica A*, 1998;261:499; Tirnakli U. and Torres D.F., *Physica A*, 1999;268:152
 - [13] Büyükkiliç F. and Demirhan D., *Euro. Phys. J. B*, 2000;14:705
 - [14] Kaniadakis G., Lavagno A. and Quarati P., *Modern Phys. Lett. B*, 1996;10:497; Kaniadakis G., Lavagno A. and Quarati P., *Nucl. Phys. A*, 2001;680:94c-97c
 - [15] Tsallis C., *Entropic nonextensivity : a possible measure of complexity*, *Chaos, Solitons & Fractals*, 2001;13:371 and e-print, cond-mat/0010150; Tsallis C., *Nonextensive statistics : theoretical, experimental and computational evidences and connections*, *Brazilian J. Physics*, 1999;29:1
 - [16] Wang Q.A. and Le Méhauté A., *Phys. Lett. A*, 1998;242:301
 - [17] Wang Q.A., Pezeril M., Nivanen L., Le Méhauté A., *Chaos, Solitons & Fractals*, 2002;13:131, e-print : cond-mat/0010294; Wang Q.A., *Chaos, Solitons & Fractals*, 2001;12:1431, e-print, cond-mat/0009343
 - [18] Lenzi E.K., Mendes R.S., da Silva L.R. and Malacarne L.C., *Physics Lett. A*, 2001;289:44
 - [19] dos Santos R.J.V., *J. Math. Phys.*, 1997;38:4104; Abe S., *Physics Lett. A*, 2000;271:74
 - [20] Nakamichi A., Joichi I., Iguchi O. and Morikava M., *Chaos, Solitons & Fractals*, 2002;13:595
 - [21] S. Abe, *Phys. Rev.E*, **63**(2001)061105; Qiuping A. Wang, Laurent Nivanen, Alain Le Méhauté and Michel Pezeril, *Note on Abe's general pseudoadditivity for nonextensive systems*, e-print : cond-mat/0111541 (submitted)
 - [22] Curilef S., *Z. Phys. B*, 1996;100:433
 - [23] Ramshaw John D., *Phys. Lett. A*, 1995;198:119; Guerberoff G.R. and Raggio G.A., *Phys. Lett. A*, 1998;214:313
 - [24] C. Anteneodo, C. Tsallis and A.S. Martinez, e-print : cond-mat/0109203
 - [25] Wang Q.A., *Unnormalized nonextensive expectation value and zeroth law of thermodynamics*, e-print : cond-mat/0111238 (submitted)
 - [26] Büyükkiliç F., H. Uncu and D. Demirhan, *Thermodynamics of the g-on gas*, eprint: cond-mat/0103560